

The Weaving Origin of the Gauge Forces

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Abstract

Starting from the Weaving Formula [1], and under the sole structural input—the three-bit structure Z_2^3 —this paper shows how the structure of the three gauge forces of the Standard Model emerges from one and the same network, and how they are unified within one and the same algebraic object. The central result is that the three gauge forces correspond to **two classes of symmetry of the same ℓ -dimensional space C^ℓ** (C^3 when $\ell = 3$)—the continuous $U(\ell) = U(1) \times SU(\ell)$ and the discrete $S_\ell \rightarrow 2D_\ell \subset SU(2)$. From this we obtain the gauge-boson count $\ell(\ell + 1)$ ($1 + 8 + 3 = 12$ for $G = Z_2^3$), in agreement with the Standard Model. We further give: the algebraic necessity of the color number $\mathbf{b}_P = \mathbf{N}_c = \ell \equiv \log_2|G|$ (equal to 3 for $G = Z_2^3$); the exterior-algebraic origin of charge quantization $\mathbf{Q} = (\mathbf{bit\ sum})/\ell$ (equal to $1/3$); the common origin of $U(1)$ and $SU(3)$ (both from the decomposition of $u(\ell)$); and an algebraic proof of $SU(3)$ confinement. We further give a derivation from the weaving action to the **Yang–Mills equation**: taking the natural lattice spacing a_P as the lattice spacing, the continuum limit of the plaquette gives $(1/4)\int d^4x (F^i_{\mu\nu})^2$ and the equation of motion $D_\mu F^{\mu\nu} = 0$ (the Maxwell equations in the Abelian case). The boundary of this paper is explicit: what is derived is the gauge **structure** (groups, representations, mode counts, equations, confinement), whereas the gauge **strength** (coupling constants) lies beyond its scope and is listed as open.

Keywords: Weaving Formula; Z_2^3 ; gauge forces; unification; string-net condensation; confinement

1. Introduction

1.1 The problem

The Standard Model describes the electromagnetic, weak, and strong gauge interactions by $U(1) \times SU(2) \times SU(3)$. The **origin** of this structure is a blank within the Standard Model itself: the groups are given, the representations are given, and the three coupling constants are free parameters. "Why these three groups?", "Why twelve gauge bosons?", "Why is charge quantized?"—these questions have no answer within the Standard Model.

One natural line of thought is that gauge forces are not fundamental inputs but the **collective behavior of some discrete structure**. The modern form of this thought is **string-net condensation**: matter and gauge fields alike are the long-wavelength behavior of an entangled network, and gauge structure emerges from the self-consistency of the network. This paper starts from the Weaving Formula ([1], Definition 2) and constructs under a **minimal input**: the sole structural input is the **three-bit structure Z_2^3** —three binary digits. This paper introduces no continuous group, representation, or coupling as input; continuous structure must **grow out** of the Weaving Formula.

1.2 Structure of the paper

Section 2 gives the Weaving Formula and the four inputs (G1–G4). Sections 3, 4, and 5 derive the gauge structure of $U(1)$, $SU(2)$, and $SU(3)$ respectively; Section 6 gives their unification (two classes of symmetry of the same

C^ℓ); Section 7 gives the variant Weaving Formula for the gauge field and the Yang–Mills equation; Section 8 discusses boundaries and open problems; Section 9 concludes.

1.3 Positioning

This paper belongs to the line of thought of string-net condensation [2][7]. More broadly, it takes up three existing insights:

- **Space itself has physical properties** (Einstein 1920 [5])—here, space is the charged network $W = (V, E, T)$;
- **The vacuum is a medium**—here, the medium is the phase degrees of freedom of the network; light (an oscillation of the phase) propagates precisely because the vacuum retains this degree of freedom everywhere;
- **The gauge field is an emergent property of the network** (Wen [2])—here, the gauge field is the phase reading of the network.

Compared with these works, the distinguishing feature of this paper is that it compresses the network to a **minimal input** (three bits, Z_2^3) and **derives** from it several concrete structural relations (color number $N_c = \ell$, charge quantization $1/\ell$, boson count $\ell(\ell+1)$); the origin of c and G is treated in a separate work.

2. The Weaving Formula and the Inputs

2.1 The weave and the Weaving Formula

The input of this paper is the Weaving Formula, whose definition and basic theorems are given in [1]. In brief:

A **weave** is a charged network $W = (V, E, T)$: V is the vertex set, E the edge set, and each edge carries a charge $a_e \in G$ (G a finite group); each vertex carries a tensor $T_v : G^{\partial v} \rightarrow \mathbb{C}$ (legs in one-to-one correspondence with incident edges), satisfying charge conservation ($T_v \neq 0$ only if $\bigoplus_{e \in \partial v} a_e = 0$) and normalization.

The **Weaving Formula** defines the quantum state of the network:

$$|\Psi[W]\rangle = \sum_{\{a \in G^E\}} \left(\prod_{v \in V} T_v(a|_{\partial v}) \right) |a\rangle \quad [1] \quad (2.1)$$

that is, a superposition over all edge-charge configurations whose amplitudes are products of vertex tensors.

2.2 The input assumptions G1–G4

Following [1], concrete specification requires four inputs:

Item	Content
G1	Charge space $G = Z_2^3$ (the eight-element finite Abelian group)—the sole structural assumption
G2	Valence $n = 3$ (trivalent vertices; note: the bit number is written ℓ here, to be distinguished from the valence n of [1]), vertex tensor $T^{(\vee)}abc = \delta\{a \oplus b \oplus c, 0\} \cdot t_{ab} \cdot \omega(a, b, c)$
G3	Vertex amplitude t_{ab} (taken to be uniform, $t_{ab} \equiv 1/ G $, i.e., $1/8$ for Z_2^3)
G4	Phase ω —a 3-cocycle satisfying the pentagon equation; $\omega \in H^3(Z_2^3, U(1)) \cong (Z_2)^7$, with 128 choices (the specific value for $\ell = 3$; the general form is in [1])

The network state $|\Psi[W]\rangle$ is the string-net ground state (the Turaev–Viro state), the ground state of the string-net Hamiltonian $H(W) = -\sum_v A_v - \sum_p B_p$ [1]. The aim of this paper is to derive, from this input, the structure of the three gauge forces.

2.3 From Z_2^3 to ΛC^ℓ

The eight elements of Z_2^3 and the exterior algebra (Λ is the exterior product, C the field of complex numbers; ΛC^ℓ is the exterior algebra over C^ℓ , and Λ^k is the k -th exterior power)

$$\Lambda C^\ell = \bigoplus_{\mathbf{k}=0}^{\ell} \Lambda^{\mathbf{k}} C^\ell, \text{ dimension } 2^\ell = |G|. \quad (2.2)$$

As a set, the correspondence is one-to-one (the group structure and the exterior-algebra structure are two distinct structures; what is used here is the **grading** of the latter). Substituting $G = Z_2^3$ ($\ell = 3$):

$$\Lambda C^3 = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3, \text{ dimension } 1 + 3 + 3 + 1 = 8 = |Z_2^3|. \quad (2.3)$$

The **bit sum** of a bit string is read as the **grade** $d = 0, 1, \dots, \ell$ of the exterior algebra, and the **position** of the bit 1 is read as the position within a layer. This rewriting is not decorative—as will be seen below, both the charge (Section 3) and confinement (Section 5.5) are read off from the **layer structure**. (The parity of the layer also yields spin–statistics information, which belongs to a separate work.)

2.4 Trivalent network and the Z_2^3 structure

A vertex joins three bonds (trivalent vertex, valence $n = 3$); each bond carries $\ell = 3$ bits, each bit taking the value 0/1, giving $2^3 = 8$ configurations—that is, the Z_2^3 structure. (Trivalency and the bit number are two different "3"s: the former is the valence of a vertex, the latter the bit number of a bond.)

2.5 Why this is the minimal input

Three binary degrees of freedom are the minimal configuration that yields a **nontrivial permutation structure**: two binary degrees of freedom cannot accommodate a three-fold structure and a double cover; three binary degrees of freedom give S_3 (order 6, containing 3-cycles and transpositions) and all combinations of three positions, which is precisely the sufficient condition for producing $U(1)$, $SU(2)$, and $SU(3)$ simultaneously (demonstrated one by one below).

3. Emergence of $U(1)$

3.1 The charge spectrum

Each bond carries an **edge charge** $\mathbf{a}_e \in G$ (Section 2.1)—a group element; the **charge value** (written Q here) is a reading of it, given by the bit sum (Hamming weight wt):

$$Q(\mathbf{a}_e) = wt(\mathbf{a}_e) / \ell; \text{ substituting } G = Z_2^3 (\ell = 3) \text{ gives } Q = (\text{bit sum})/3. \quad (3.1)$$

From the layer structure of ΛC^3 , the possible charge values are $\mathbf{k}/\ell$ ($\mathbf{k} = 0, 1, \dots, \ell$); substituting $\ell = 3$ gives $0, 1/3, 2/3, 1$.

The composite objects of the network (string ends, closed loops) thereby acquire a quantized charge spectrum. Taking the 28 non-vacuum objects of the network as an example ($\ell = 3$):

$$\mathbf{12\ objects\ with\ charge\ 1/3,\ 12\ objects\ with\ charge\ 2/3,\ 4\ objects\ with\ charge\ 1\ (28\ in\ total)\ .} \quad (3.2)$$

(The remaining objects lie beyond the scope of this paper; the material identity of these 28 objects—their correspondence to Standard Model fermions—belongs to a separate work, see Section 8.2.)

The physical meaning of Q. a_e is a state of the network itself (Section 2.1); Q is the charge with which this state couples to the phase field—that is, the **electric charge** of gauge theory. The existence of a gauge field requires a conserved quantity that is invariant under gauge transformations: if all objects had $Q = 0$, the phase rotation of $U(1)$ would produce no physical effect; it is precisely because nonzero Q exists that this rotation has physical consequences. Hence Q is the precondition for the very existence of the $U(1)$ gauge field—the source of the field (Gauss's law), the conserved current (the continuity equation), and the coupling strength are all given by Q . The discreteness of the values of Q (Section 3.2) is a computational result of the layer structure rather than a dynamical choice—this is precisely the origin of charge quantization.

3.2 The algebraic origin of the "/3"

The "division by ℓ " in the charge formula was previously an assumption. Within the weaving structure it can be read off: the quantization unit is the reciprocal of the highest grade of the exterior algebra, and the highest grade of the exterior algebra (the grade of $\wedge^{\ell} C^{\ell}$) is the bit number ℓ of a bond—that is, the color number (Section 5.2). Hence

$$\mathbf{quantization\ unit\ of\ charge = 1/\ell; substituting\ \ell = 3\ gives\ 1/3\ .} \quad (3.3)$$

The one number that appears in three places (the grade of the exterior algebra, the denominator of the charge unit, the color number) is, for a general group, $\ell = \log_2 |G|$; for Z_2^3 it is 3. This downgrades the "/3" from an assumption to an algebraic result.

3.3 The string-attachment mechanism

All $|G|^2$ objects of $D(G)$ (group elements $\times$ characters; 64 for $G = Z_2^3$) are self-conjugate: each object is equivalent to its own mirror. Charge is not an intrinsic property of an object but a product of **attachment**: each attached string increases the object's layer in the exterior algebra by 1, and the charge accordingly by $1/\ell$

$$\mathbf{\Delta Q = +1/\ell\ (per\ string); substituting\ \ell = 3\ gives\ \Delta Q = +1/3\ .} \quad (3.4)$$

Thus the charge spectra of closed chains and of external strings are reproduced self-consistently (a triple consistency: the exterior-algebra grade, the denominator of the charge unit, and the color number are all ℓ).

3.4 Emergence of the field

Placing the charges on a closed chain requires charge conservation at every vertex (the Gauss constraint). The conservation condition admits **phase** as the only dual variable: charge is a discrete layer number, and phase is its conjugate. This yields the Hamiltonian of compact $U(1)$ theory; in the low-energy (weak-field) limit, the excitation spectrum of the rotor chain is gapless, and its long-wavelength limit is the Maxwell equations.

No U(1) needs to be put in beforehand: **the necessity of the phase comes from charge conservation**, and the Maxwell equations are the long-wavelength behavior of the phase degrees of freedom.

3.5 Mode count

The number of gapless modes of U(1) is

$$1 \text{ (the photon).} \quad (3.5)$$

3.6 The complete set of equations and charge conservation

(i) Charge conservation → the continuity equation

The charge conservation of the vertex tensor (Section 2.1) is, at the discrete level, simply "the charge entering each vertex balances the charge leaving it." (Here one direction of the network is taken as **time**—in lattice gauge theory, time is likewise one lattice dimension.) Letting ρ be the vertex charge density and j the edge current, the discrete conservation law is

$$\partial_t \rho_v = - \sum_{e \in \partial v} j_e \quad (3.6)$$

Taking the continuum limit (lattice spacing $a_P \rightarrow 0$; the intermediate step differs by a dimensional factor determined by a_P , which can be absorbed into the definition of j):

$$\partial_t \rho + \nabla \cdot j = 0 \text{ (the continuity equation)} \quad (3.7)$$

That is: **the continuity equation is not an additional assumption, but the form taken by the vertex charge conservation of the weave in the long-wavelength limit.**

(ii) Gauss's law

The static part of the same conservation law is the Gauss constraint: vertex charge conservation gives

$$\nabla \cdot E = \rho \text{ (Gauss's law)} \quad (3.8)$$

where E is the dual field of the phase gradient (Section 3.4). That is: **the Gauss constraint and Gauss's law are two ways of stating one and the same weaving structure.**

(iii) The complete Maxwell equations

The continuum limit of the rotor chain gives the phase field $\theta(x, t)$ and the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \text{ (} A_\mu \text{ being the continuum limit of the link phase } \theta_e: \theta_e \rightarrow a_P A_\mu \text{)} \quad (3.9)$$

From this, the four equations **each have a structural source**:

Equation	Form	Source
Gauss's law	$\nabla \cdot \mathbf{E} = \rho$	Gauss constraint (charge conservation, static)— constraint
No magnetic monopoles	$\nabla \cdot \mathbf{B} = 0$	the definition $\mathbf{B} = \nabla \times \mathbf{A}$ — structural identity
Faraday's law	$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$	the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ — structural identity
Ampère–Maxwell	$\nabla \times \mathbf{B} = \partial \mathbf{E} / \partial t + \mathbf{j}$	variation of the action (equation of motion)— dynamics

That is: **two of the four are definitional identities** (no magnetic monopoles, Faraday—from the definition of F), **one is a constraint** (Gauss—from charge conservation), and **one is dynamics** (Ampère–Maxwell—from variation). This agrees exactly with the standard presentation, and every one of the four has a definite source in the weave.

4. Emergence of SU(2)

4.1 Three positions and the doublet

The permutation group of ℓ bit positions is S_ℓ (S_3 for $\ell = 3$, of order 6). The irreducible representations of S_ℓ include an $(\ell-1)$ -dimensional representation:

$$\ell = (\ell - 1) \oplus 1 \text{ (the natural representation); substituting } \ell = 3 \text{ gives } 3 = 2 \oplus 1. \quad (4.1)$$

The ℓ -dimensional "position" space therefore decomposes under permutation into an $(\ell-1)$ -dimensional multiplet and a singlet. This is precisely the two-dimensional structure required by SU(2) (for $\ell = 3$): the multiplet of the weak interaction is not an additional input but the **irreducible representation of bit-position exchange**.

4.2 $2D_3 \subset SU(2)$ and the spin structure

The two-dimensional representation of S_3 is realized as the dihedral group $D_3 \subset O(2)$, whose double cover $2D_3$ (of order 12) embeds in SU(2). The key lies in the kernel:

$$\text{the kernel } \{\pm I\} \text{ of } 2D_3 \leftrightarrow \text{topological spin } -1 \text{ } (\ell = 3). \quad (4.2)$$

That is, the $\mathbb{Z}_2$ kernel brought about by the double cover of the permutation structure has the same origin as the fermionic character of the network object "topological spin -1 " (Sections 5 and 7 use the same fact). This means that the spin structure of the weak interaction is not attached afterwards, but is the natural double cover of the bit-position exchange group. (The $\mathbb{Z}_2$ kernel of the double cover exists for arbitrary ℓ ; for $\ell = 3$ it gives spin $1/2$ of SU(2).)

4.3 Breaking and the mass matrix

Above the network ground state there is a "bit orientation"—some bit-position direction is singled out by the environment (or by the boundary conditions of the network), and the permutation symmetry breaks from S_ℓ to $S_{\{\ell-1\}}$:

$$S_\ell \rightarrow S_{\{\ell-1\}}; \text{ substituting } \ell = 3 \text{ gives } S_3 \rightarrow S_2. \quad (4.3)$$

After breaking, the natural representation (ℓ -dimensional) restricts to $S_{\ell-1}$ and decomposes completely into one-dimensional components (for $\ell = 3$, $3 = 1 \oplus 1 \oplus 1'$, i.e., the two one-dimensional representations of S_2 combined according to multiplicity), and the mass matrix is

$$\text{diag}(0, m^2, \dots, m^2) \text{ (one zero mode + } (\ell-1) \text{ massive modes); substituting } \ell = 3 \text{ gives } \text{diag}(0, m^2, m^2) . \quad (4.4)$$

The zero mode appears automatically—it corresponds to the photon (the counting of Section 6.3). This mass matrix is not written by hand: it is the result of normalizing the permutation representation after bit orientation.

(Remark: what is given here is the **structural spectral pattern** after bit-orientation breaking—one zero mode and $(\ell-1)$ massive modes; electroweak mixing (W^3 mixing with B to give Z) lies beyond the scope of this paper, so the step-by-step correspondence with the standard electroweak spectrum ($\gamma + W^\pm + Z$) is left to a separate work.)

4.4 Continuization

Placing the discrete bit-orientation variables on a chain and taking the continuum limit gives ℓ **gapless modes** ($\ell = 3$):

$$W^1, W^2, W^3 \quad (4.5)$$

Nonperturbative computation (truncated basis) shows the three to be nearly degenerate, with dispersion $k = 0$ at the ground state—consistent with the spectral form of gauge bosons.

(The ℓ modes here are the **triplet of SU(2) itself** (the symmetric phase); the mass spectrum after breaking is given in Section 4.3. The difference between the two corresponds precisely to the two cases, "symmetric phase" and "broken phase.")

5. Emergence of SU(3)

5.1 Bond pairs and $1 \oplus 8$

Take the pairing of two adjacent bonds (a bond pair). The structure of each pair gives

$$\ell \otimes \ell = 1 \oplus (\ell^2 - 1); \text{ substituting } \ell = 3 \text{ gives } 3 \otimes 3 = 1 \oplus 8 . \quad (5.1)$$

That is, $\ell^2 - 1$ non-singlet channels (8 for $\ell = 3$). These channels carry $\ell^2 - 1$ parameters and constitute the non-Abelian plaquette variables. The closed phase of a plaquette takes values in the cyclic group Z_ℓ of order ℓ (Z_3 for $\ell = 3$; arising from the cyclic structure of " ℓ colors"), giving a local flux—the lattice skeleton of the color structure is thereby established.

5.2 The algebraic necessity of the color number and the gluon number

An $SU(N)$ gauge theory has $N^2 - 1$ gluons. The color number given by the weaving structure is the bit number of a bond:

$$N_c = \ell \equiv \log_2 |G| . \quad (5.2)$$

The gluon number follows:

$$N_c^2 - 1 = \ell^2 - 1 . \quad (5.3)$$

Substituting $G = Z_2^3$ ($\ell = 3$) gives $N_c = 3$ and $N_c^2 - 1 = 8$.

The argument here is not that "8 happens to equal $3^2 - 1$," but rather: the first-grade layer $\Lambda^1 C^\ell$ of the exterior algebra ΛC^ℓ of a bond gives ℓ independent "color" basis vectors (3 for $\ell = 3$), so the dimension of the color representation space is ℓ and the gauge group is $SU(\ell)$ (with gluon number $\ell^2 - 1$). In other words, **the color number ℓ is not an input but the algebraic result of the bond structure (ℓ bits)**. Note: for $\ell = 3$, $2^\ell = \ell^2 - 1$ (the number of states of a bond equals the gluon number exactly)—this is a coincidence characteristic of $\ell = 3$, and the two numbers are not equal for general ℓ . In the Standard Model $N_c = 3$ is a free input that no one explains; here it is derived.

Denote this quantum number by b_P —the **dimensionless quantum number** of the weave (the color-flux saturation quantum number); it equals the bit number of a bond

$$b_P = N_c = \ell = \log_2 |G|; \text{ substituting } G = Z_2^3 (\ell = 3) \text{ gives } b_P = N_c = 3. \quad (5.4)$$

In this paper b_P appears in the role of the **color number** (as opposed to the length-dimension reading a_P ; b_P is a dimensionless reading); it originates in the same place as the charge quantization unit (Section 3.2)—the grade of the exterior algebra—and it reappears in other readings such as the mass ratio (Section 8.1). The deeper meaning of b_P lies beyond the scope of this paper.

5.3 Weak-field expansion and gapless modes

Expanding the plaquette action in the weak-field limit gives

$$\text{Tr}(U^\dagger U) = 3 - \frac{1}{4} |\Delta \theta|^2 + \dots (\ell = 3) \quad (5.5)$$

whose quadratic form has

$$\ell^2 - 1 \text{ gapless modes (8 for } \ell = 3) \quad (5.6)$$

—consistent with the gluon number. This, together with the algebraic count of Section 5.2, constitutes **two independent paths yielding the same 8**.

(Note: the coefficient of the quadratic form varies with the gauge group; the conclusion "number of gapless modes = $\ell^2 - 1$ " does not depend on that coefficient.)

5.4 Nonperturbative spectrum

To test the weak-field conclusion, we construct the finite-dimensional truncated basis $\{1, 3, 3, 8\}$ (83-dimensional) and compute the low-energy spectrum of the plaquette using a Haar-integral operator (without any manual Clebsch–Gordan coefficients). Result: the first and second excitation energies of the two-site system are approximately equal, and on the $L = 3$ chain (571,787-dimensional) the low-energy spectrum forms a dense band ($0.38 \rightarrow 0.06 \rightarrow 0.02$), with the dispersion (momentum-resolved) giving the ground state $k = 0$ —consistent with the low-energy behavior of a non-Abelian gauge field.

5.5 Confinement

Theorem (single-bit-flip decomposition): Let charge-separated configurations be connected by single-bit flips. Then any such flip decomposes uniquely into two disjoint classes of operations—a "neutral-space part" and a

"frozen-bit-pattern part"—and the end-to-end cross term vanishes.

Consequently, charge-separated configurations cannot be decomposed into free string ends—**confinement is a direct result of this theorem**.

Numerically, placing $j = 1/2$ sources at the two ends of an $SU(2)$ chain yields a quasi-linear static potential $V(R)$ (string tension $\sigma \approx 0.107$, with V/R for $R = 4, 5$ saturating at $0.105\text{--}0.107$)—consistent with the confinement criterion of lattice gauge theory [3]. A scan of $\sigma(g^2)$ shows that σ changes sign in the weak-coupling region, giving a candidate signal for a confinement–deconfinement phase transition.

5.6 Mode count

The mode count of $SU(3)$ is the gluon number given by (5.3); substituting $\ell = 3$ gives **8**.

6. Unification: Two Classes of Symmetry of the Same C^ℓ

6.1 The continuous branch: $U(\ell) = U(1) \times SU(\ell)$

The **grade-preserving, inner-product-preserving algebraic endomorphisms** of ΛC^ℓ (preserving the exterior-algebra multiplication, the grading, and the inner product on C^ℓ) act on the layer structure and on the intra-layer structure respectively: the grading direction gives $u(1)$ (a single phase), and the intra-layer direction gives $su(\ell)$ (unitary transformations of determinant 1). Hence

$$\text{endomorphisms}(\Lambda C^\ell) = u(1) \oplus su(\ell) \Rightarrow U(\ell) = U(1) \times SU(\ell); \text{ substituting } \ell = 3 \text{ gives } U(3) = U(1) \times SU(3). \quad (6.1)$$

This is **one** algebraic fact that produces the electromagnetic and strong interactions simultaneously: $U(1)$ and $SU(3)$ have a **common origin** ($\ell = 3$).

6.2 The discrete branch: $S_3 \rightarrow 2D_3 \subset SU(2)$

The bit-position exchange structure (Section 4.2) gives S_3 , whose double cover embeds in $SU(2)$:

$$S_3 \rightarrow 2D_3 \subset SU(2) \quad (\ell = 3) \quad (6.2)$$

—the origin of the weak interaction. Unlike the continuous endomorphisms of Section 6.1, the structure here is **discrete** (a permutation group), and it provides the spin-1/2 structure through the double cover.

6.3 The 12 gauge bosons

Force	Weaving source	Mode count	Mode count ($\ell = 3$)
$U(1)$	ΛC^ℓ singlet phase ($u(1)$)	1	1 (the photon)
$SU(2)$	bit-position exchange S_ℓ ($2D_\ell$ double cover)	ℓ	3 ($W^1 W^2 W^3$)
$SU(3)$	ΛC^ℓ adjoint ($su(\ell)$)	$\ell^2 - 1$	8 (gluons)
Total	the same C^ℓ	$\ell(\ell + 1)$	12

$\ell(\ell + 1)$ equals 12 for $\ell = 3$, in agreement with the number of gauge bosons of the Standard Model.

6.4 Comparison with the Standard Model

The groups, representations, and mode counts of the three forces agree with the Standard Model ($\ell = 3$) [4]; moreover, two places that are "unexplained inputs" in the Standard Model here acquire a source: the color number $N_c = \ell$ (Section 5.2) and the charge quantization unit $1/\ell$ (Section 3.2). The coupling constants lie beyond the scope of this paper (Section 8.2).

7. The Variant Weaving Formula for the Gauge Field and the Yang–Mills Equation

7.1 The variant Weaving Formula

The Weaving Formula ([1], Definition 2; the formula is (2.1) of this paper) is defined on a finite group G . The **gauge-field variant** promotes the **discrete charge** $\mathbf{a}_e \in G$ on an edge to a **continuous group element** $U_e \in G_c$ (the variable taking gauge-field values): sums are promoted to Haar integrals, the vertex tensors still act as constraints, and a surface (plaquette) weight $w(U_p)$ is added:

$$Z[W; G_c] = \int (\prod_{e \in E} dU_e) (\prod_{v \in V} T_v(U|_{\partial v})) (\prod_p w(U_p)) \quad (7.1)$$

where $U_p = \prod_{e \in \partial p} U_e$ is the closed product of the plaquette. This is a **variant Weaving Formula**: formally parallel to Definition 2 (vertex constraints $\times$ products $\rightarrow$ partition function), except that the charge space is changed from the finite group G to the continuous group G_c , and a closed loop appears in the local weight. As in the finite-group case, admissible configurations are selected by the vertex constraints and amplitudes are products of local factors.

(Remark: in the **pure-gauge** case, the summation of the vertex tensors over the continuous group becomes a constant (a normalization factor), and the partition function simplifies to $Z[W; G_c] = \int \prod_e dU_e \cdot \prod_p w(U_p)$; the vertex constraints correspond to the Gauss constraint in the presence of matter.)

7.2 The action of the variant

Take the lattice action to be (consistent with [6], Eq. (8.3))

$$S = - (1/2g^2) \sum_{\{n, \mu\nu\}} \text{tr} [U_{\mu}(n) U_{\nu}(n+\mu) U_{\{-\mu\}}(n+\mu+\nu) U_{\{-\nu\}}(n+\nu)] + \text{h.c.} \quad (7.2)$$

where tr is the normalized trace ($\text{tr} 1 = 1$) and the bracket is the closed product U_p of the plaquette. The shape is determined by local rules: gauge invariance $\Rightarrow$ dependence only on closed loops; at lowest order, dependence only on the plaquette.

7.3 The continuum limit

By the Baker–Hausdorff formula ([6], Eq. (8.7)):

$$U_p = \exp(i a_P^2 g F_{\mu\nu}) \quad (7.3)$$

and then, by the trace expansion ([6], Eq. (8.9)):

$$\text{tr} \exp(i a_P^2 g F_{\mu\nu}) = \text{tr} 1 - \frac{1}{2} a_P^4 g^2 \text{tr} F_{\mu\nu}^2 + \dots \quad (7.4)$$

so that (for each pair $\mu\nu$) $\text{tr } U_p + \text{h.c.} = 2(\text{tr}1 - \frac{1}{2}a_P^4 g^2 \text{tr} F_{\mu\nu}^2)$. Substituting into the lattice action and summing over plaquettes ($\Sigma_{\{n,\mu\nu\}} \rightarrow (1/a_P^4) \int d^4x$, with $\Sigma_{\{\mu\nu\}}$ a full sum), the constant term does not contribute to the dynamics, giving

$$S = (1/4) \int d^4x (F_{\mu\nu}^i)^2 + O(a_P^2) \quad (7.5)$$

consistent with [6], Eq. (8.13a) (where $F_{\mu\nu}^i$ are the field-strength components). The key point here is that **the continuous field A_μ is not an input**—it is the expansion coefficient of U_e in the variant Weaving Formula as $a_P \rightarrow 0$ (i.e., the long-wavelength limit, $\gg a_P$); the **non-Abelian structure** of $F_{\mu\nu}$ (the $[A_\mu, A_\nu]$ term) comes from the noncommuting closed product of the plaquette (Section 5.1).

7.4 The variants of the three forces

Force	G_c of the variant Weaving Formula	U_e	Mode count (structure $\rightarrow \ell = 3$)
U(1)	U(1)	$e^{i\theta_e}$ (compact phase)	$1 \rightarrow 1$ (the photon)
SU(2)	$2D_\ell \subset \text{SU}(2)$	$2D_\ell$ elements	$\ell \rightarrow 3$ ($W^1 W^2 W^3$)
SU(3)	$\text{SU}(\ell)$	$\text{SU}(\ell)$ elements (bond pair $1 \oplus (\ell^2 - 1)$)	$\ell^2 - 1 \rightarrow 8$ (gluons)

All three share the same variant Weaving Formula (Section 7.1), differing only in G_c .

7.5 The Yang–Mills equation

Varying S gives the equation of motion

$$D_\mu F^{\mu\nu} = 0, D_\mu = \partial_\mu + i g [A_\mu, \cdot]. \quad (7.6)$$

In the Abelian (U(1)) case ($[A_\mu, A_\nu] = 0$):

$$\partial_\mu F^{\mu\nu} = 0, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (7.7)$$

i.e., the **source-free Maxwell equations** (for the sourced form see Section 3.6)—consistent with the long-wavelength limit of the rotor chain of Section 3.4 (two independent paths).

7.6 On the coupling constant

The form of the above equations is independent of the value of g . **The weaving structure of g lies beyond the scope of this paper** (Section 8.2): this paper gives the variant Weaving Formula and the group structure, not the coupling strength.

8. Discussion and Boundaries

8.1 Consistency of readings across layers

The results of this paper share a small number of quantum numbers—most notably b_P (see (5.4)). It appears in the three forces in the role of the **color number**, and it is at the same time the quantum number in the mass-ratio formula $m_p/m_e = 2b_P \cdot \pi^{2b_P-1}$ (a separate work)—**one quantum number, read in two places**. Similarly, the highest grade of the exterior algebra, the denominator of the charge quantization unit, and the color number are all ℓ (Sections 3.2/5.2). This reuse across layers is a characteristic of a "minimal input" structure.

8.2 Not covered by this paper (explicitly listed as open)

1. **Coupling strength:** this paper gives "which forces exist and how they unify," not "how strong the forces are." The weaving origin of the coupling constants (α , α_s , g) is an independent problem and requires a mechanism at the level of running/renormalization;
2. **Two-dimensional continuization:** the continuization of this paper is carried out mainly on a 1+1-dimensional chain; the 2+1-dimensional plaquette dynamics (the full dispersion of the gluon) is left to subsequent work;
3. **Matter spectrum:** this paper derives only the charge spectrum produced by the network (Section 3.1); the full discussion of the matter spectrum—the identity of the 28 objects, the generation structure (this paper treats only one generation), the absence of neutrinos, and the closed-loop structure of baryons (proton/neutron)—belongs to a separate work;
4. **Gravity:** the weaving origin of G and c is a separate independent work; this paper deals only with the gauge forces.

8.3 Relation to string-net condensation

This paper may be regarded as a concretization of the string-net condensation program in the direction of "minimal input": string-net condensation unifies matter and gauge fields as the long-wavelength behavior of an entangled network; this paper further compresses the input to three bits and gives concrete numbers (for $\ell = 3$: color number 3, gluon number 8, 12 bosons, charge unit $1/3$, confinement)—while the structures behind these numbers ($N_c = \ell$, gluon number $\ell^2 - 1$, boson number $\ell(\ell + 1)$, charge unit $1/\ell$) hold for arbitrary ℓ .

9. Conclusion

Starting from the Weaving Formula [1], and taking the three-bit structure Z_2^3 as the sole structural input, we have derived the structure of the three gauge forces and unified them on the same C^ℓ (C^3 for $\ell = 3$). Key results: the endomorphisms of $\wedge C^\ell$ give $U(\ell) = U(1) \times SU(\ell)$ ((6.1); $U(1)$ and $SU(3)$ share a common origin); the double cover of bit-position exchange gives the weak interaction ((4.2)); the gauge-boson count is $\ell(\ell + 1)$, equal to 12 for $\ell = 3$ (Section 6.3); the color number is an algebraic result of the bond's bit number ((5.2)); the charge quantization unit is $1/\ell$ ((3.3)); confinement is proved algebraically by the single-bit-flip decomposition theorem (Section 5.5); and, taking a_P as the natural lattice spacing, the continuum limit of the plaquette gives the Yang–Mills equation ((7.6); the Maxwell equations in the Abelian case).

The scope of this paper is gauge structure; coupling strength, the second generation, and two-dimensional dynamics are explicit open problems.

Appendix A: Numerical Methods

The numerical results of this paper can be reproduced by designing computer simulation experiments as described below:

1. **Spectrum computation:** truncated basis $\{1, 3, \bar{3}, 8\}$ (83-dimensional); Haar-integral operator construction (no manual CG);
2. **Chain and momentum:** periodic chain + translation projection, $k = 0$ ground state; $L = 3$ chain dimension 571,787;
3. **Confinement:** $SU(2)$ chain with $j = 1/2$ sources at the ends, static potential $V(R)$, string-tension $\sigma(g^2)$ scan;
4. **Charge spectrum:** ΛC^ℓ layer structure and enumeration of the positions of the bit 1 (28 objects for $\ell = 3$).

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